# Modelling neutron star mountains

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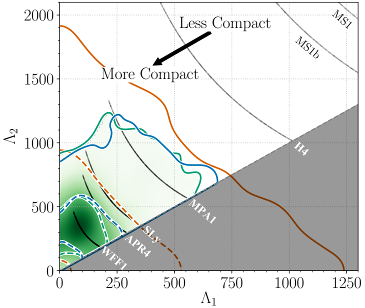
# Overview

# Introduction

# Neutron stars as gravitational-wave sources

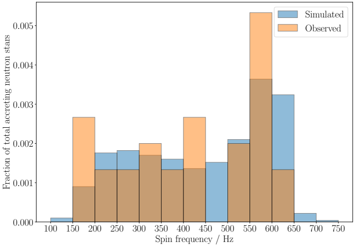
- With the first detections binary neutron star mergers (GW170817 and GW190425) we know that these systems are promising gravitational-wave sources.
- There are a variety of mechanisms through which neutron stars can radiate gravitational waves. These include:
  (i) binary inspiral and merger (Abadie *et al.*, 2010);
  (ii) modes of oscillation and their corresponding instabilities (Friedman and Schutz, 1978); and
  (iii) rotating neutron stars that host (non-axially symmetric) deformations known as *mountains* (Bildsten, 1998).




Marginalised posterior for the tidal deformabilities of the two binary components of GW170817 (Abbott *et al.*, 2018).

# Rapidly rotating pulsars

- No neutron star has been observed that rotates (even remotely) close to the centrifugal break-up frequency (∼ 1 kHz for most equation-of-state candidates).
- There are two competing explanations for this behaviour:
  1. An interaction between the accreting gas and the magnetic field lines.
  2. Gravitational-wave emission.
- Searches for evidence of rotating neutron stars in gravitational-wave data have, so far, only provided upper limits on the size of the deformations.




The spin distribution of accreting pulsars with millisecond periods. The observed population is shown in blue and a simulated population of transiently accreting systems with $Q_{22} = 10^{36}\,\mathrm{g\,cm^2}$ is shown in orange (Gittins and Andersson, 2019).

## Maximum mountain calculations

# The quadrupole moment

- When a star is deformed away from perfect sphericity it develops multipole moments, $Q_{lm}$. In the Newtonian limit, these are (usually) defined as

$$Q_{lm} \equiv \int_0^R \delta\rho_{lm}(r) r^{l+2} \, dr.$$

- Equivalently, one may read off the multipole moments from the exterior potential,

$$\delta\Phi_{lm}(r) = -\frac{4\pi G}{2l+1} \frac{Q_{lm}}{r^{l+1}} \qquad \text{for } r \geq R.$$

- The dominant multipole in gravitational-wave emission is the quadrupole, $Q_{22}$. Thus, we will specialise to the $(l, m) = (2, 2)$ mode.
- It is common, in observation papers, to use the *fiducial ellipticity*,

$$\epsilon = \sqrt{\frac{8\pi}{15}} \frac{Q_{22}}{I_{zz}},$$

where $I_{zz}$ is the principal stellar moment of inertia (which we take to have the fiducial value of $I_{zz} = 10^{45}\,\mathrm{g\,cm^2}$).

# Stellar models

- One can use the equation of force balance – the Euler equation – to characterise a stellar model,

$$\begin{aligned} 0 &= -\nabla_i p - \rho \nabla_i \Phi + \nabla^j t_{ij} + f_i \\ &\equiv -H_i + \nabla^j t_{ij} + f_i. \end{aligned}$$

  where $t_{ij}$ is the trace-free shear-stress tensor for an elastic solid and $f_i$ is a deforming force. We will consider perturbations of this equation with respect to different stellar models.

- The following boundary conditions must be satisfied:

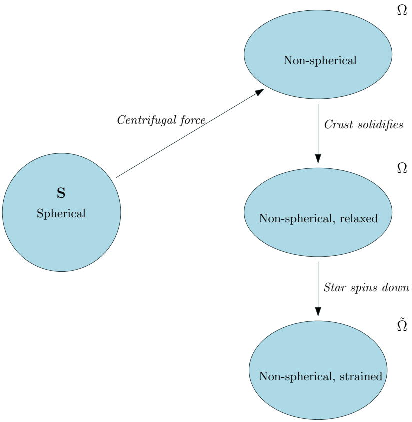
$$\delta\Phi(0) = 0, \tag{1a}$$

$$R\delta\Phi'(R) = -(l+1)\delta\Phi(R) \tag{1b}$$

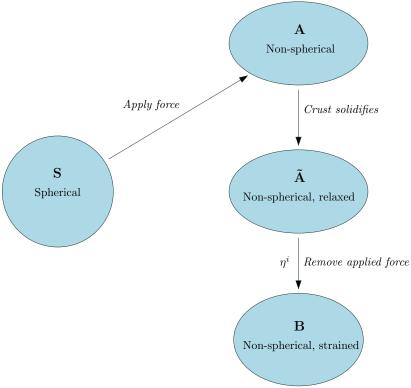
  and the traction, $(p\, g_{ij} - t_{ij})\nabla^j r$, must be continuous throughout the star.

# A rotation example

- Suppose a young star with a molten crust spins at an angular frequency, $\Omega$. (Such a star may be constructed by incorporating the centrifugal force into the Euler equation.)
- At this rotation rate, the star cools and the crust solidifies.
- The star then spins down to a frequency, $\tilde{\Omega} < \Omega$. Because it has spun down, the shape of the star changes, which builds up strain in the crust.



# Building mountains: the usual approach I




- **Star S**:

$$H_i^{\mathrm{S}} = 0. \tag{2}$$

- **Star A**:

$$H_i^{\mathrm{A}} = f_i. \tag{3}$$

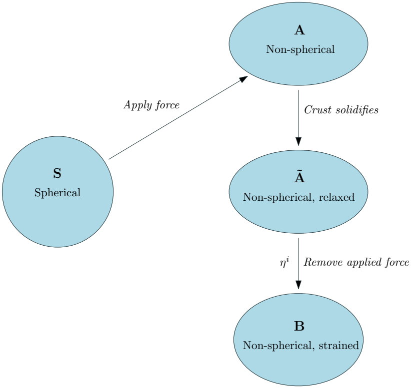
- **Star Ã**:

$$H_i^{\tilde{\mathrm{A}}} = H_i^{\mathrm{A}} = f_i. \tag{4}$$

- **Star B**:

$$H_i^{\mathrm{B}} = \nabla^j t_{ij}(\eta). \tag{5}$$

- It is star B that we are ultimately interested in. It is the star with a mountain that is supported by elastic stresses.




- The difference between stars B (5) and S (2) is simply

$$\delta H_i^{\mathrm{SB}} \equiv H_i^{\mathrm{B}} - H_i^{\mathrm{S}} = \nabla^j t_{ij}(\eta). \quad (6)$$

  This is the expression that we wish to evaluate to obtain the quadrupole.

- This equation immediately implies a procedure for calculating the perturbations: suppose one supplies a strain field, $\sigma_{ij} = t_{ij}/(2\mu)$, then one can solve for the perturbations on the left-hand side of (6).

# Ushomirsky, Cutler, and Bildsten I

- The earliest calculation of the maximum mountain a neutron star crust can support was conducted by Ushomirsky, Cutler, and Bildsten (2000, hereafter UCB). They worked in Newtonian gravity with the Cowling approximation, $\delta\Phi = 0$.
- UCB used the perturbed Euler equation (6) to obtain an integral expression for the quadrupole in terms of the shear stresses,

$$Q_{22} = \int_{r_\mathrm{b}}^{r_\mathrm{t}} \frac{r^4}{\Phi'} \left[ \frac{3}{2} t'_{rr} - \frac{4}{\beta} t'_{r\perp} - \frac{r}{\beta} t''_{r\perp} + \left( \frac{1}{2} - \frac{1}{\beta^2} \right) t'_\Lambda + \frac{3}{r} t_{rr} - \frac{\beta}{r} t_{r\perp} \right] dr.$$

- UCB conjectured that the star would attain its maximum quadrupole when the crust was maximally strained [where all the strain was assumed to be in the $(l, m) = (2, 2)$ mode] *at every point*. For a $1.4\,M_\odot$, 10 km star, they found

$$Q_{22}^{\max} \approx 1.2 \times 10^{39} \left( \frac{\bar{\sigma}_{\max}}{10^{-1}} \right) \ \mathrm{g\,cm}^2, \quad \epsilon^{\max} \approx 1.6 \times 10^{-6} \left( \frac{\bar{\sigma}_{\max}}{10^{-1}} \right).$$

# Ushomirsky, Cutler, and Bildsten II

- However, the UCB strain components do not respect the continuity of the perturbed traction, $(\delta p\, g_{ij} - t_{ij})\nabla^j r$, when the shear modulus, $\mu$, is discontinuous (*e.g.*, at a phase transition).
- One can show, in the case when $\sigma_\Lambda = \text{const}$, that for the traction conditions to be satisfied, one must have a strain with

$$\sigma_{r\perp} = 0$$

and

$$\frac{2}{\beta} r\sigma'_{r\perp} = 3\sigma_{rr} + \left(1 - \frac{2}{\beta^2}\right)\sigma_\Lambda$$

at the crustal boundaries.

- Furthermore, if $\mu$ is assumed to smoothly go to zero at the crust boundaries, then one does not have enough equations to uniquely determine the displacement.

- Haskell, Jones, and Andersson (2006, hereafter HJA) set out to relax some of the assumptions made by UCB. This included dropping the Cowling approximation and ensuring the continuity of the traction.
- HJA did not specify a strain field. They derived the perturbation equations with respect to a spherical background and increased the perturbation amplitude until the crust began to break *at any point*. They obtained results approximately an order of magnitude larger than UCB,

$$Q_{22}^{\max} \approx 3.1 \times 10^{40} \left( \frac{\bar{\sigma}_{\max}}{10^{-1}} \right) \text{ g cm}^2, \quad \epsilon^{\max} \approx 4.0 \times 10^{-5} \left( \frac{\bar{\sigma}_{\max}}{10^{-1}} \right).$$

# Haskell, Jones, and Andersson II

- Although HJA correctly dealt with the traction, they incorrectly (implicitly) assumed the relaxed shape to be spherical.
- Additionally, they introduced an additional force at the surface which ensured the star was deformed in an $(l, m) = (2, 2)$ way. Because the mountains are sustained by this surface force, the maximum quadrupoles calculated using this framework are insensitive to the shear modulus of the crust.
- Without the deforming force, their formalism does not have the freedom to satisfy both boundary conditions (1) on the perturbed potential, $\delta\Phi$.
- Finally, there are typographical errors in the perturbation equations which, once corrected, increase the mountain size by three orders of magnitude, highlighting the conceptual problem with this approach.

# Johnson-McDaniel and Owen

- The most recent maximum mountain estimates come from Johnson-McDaniel and Owen (2013, hereafter JMO). They generalised the UCB approach to relativistic gravity while relaxing the Cowling approximation. They calculated, for a $1.4\,M_\odot$ star,

$$Q_{22}^{\max} \approx 2 \times 10^{39} \left( \frac{\bar{\sigma}_{\max}}{10^{-1}} \right) \text{ g cm}^2, \quad \epsilon^{\max} \approx 3 \times 10^{-6} \left( \frac{\bar{\sigma}_{\max}}{10^{-1}} \right).$$

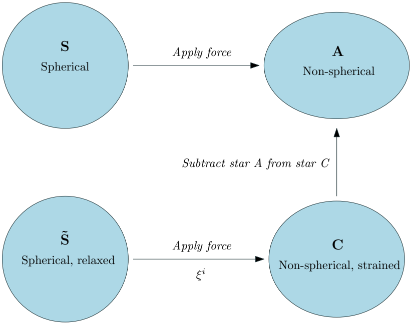
- For the same reasons the UCB approach does not obey the traction conditions at the crustal boundaries, this calculation also does not keep the traction continuous throughout the star.
- In addition, the perturbed stress-energy tensor used by JMO does not include variations of the four-velocity.

## A new mountain scheme

# A new mountain scheme I

- It would seem that calculating mountains in the usual fashion, via the perturbed Euler equation (6), has some complications:
  1. One could specify a strain field, but care must be taken in ensuring that it satisfies the traction conditions at the boundaries.
  2. One could amend the approach of HJA so that it assumed a non-spherical shape for the relaxed configuration, but one would need to specify the exact shape.
- To this end, we introduce another scheme which makes explicit use of the deforming force.

# A new mountain scheme II




- **Star $\tilde{\text{S}}$**:

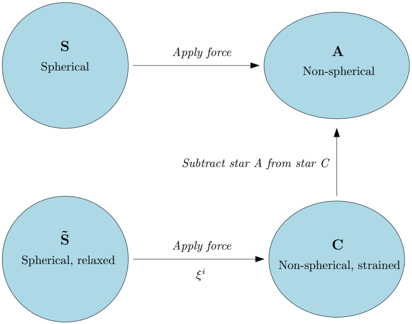
$$H_i^{\tilde{\text{S}}} = H_i^{\text{S}} = 0. \tag{7}$$

- **Star C**:

$$H_i^{\text{C}} = \nabla^j t_{ij}(\xi) + f_i. \tag{8}$$

- The force, $f_i$, is a proxy for the formation history of the star that results in its non-spherical, relaxed shape (that may involve complex processes like plastic flow and cracking). There is no requirement that this force ever acted on the star.

## A new mountain scheme III

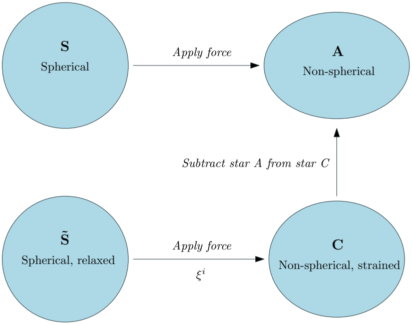



- The difference between stars C (8) and A (3) gives

$$\delta H_i^{\mathrm{AC}} = \nabla^j t_{ij}(\xi). \qquad (9)$$

- The mountains calculated using (9) are equivalent to solving the previous perturbed Euler equation (6).
- Using the deforming force, $f_i$, has two main advantages: (i) we can calculate the relaxed configuration and (ii) the explicit introduction of the force into the Euler equation provides the necessary freedom to calculate the displacement and satisfy all the boundary conditions.

# A new mountain scheme IV




- We can use this scheme to calculate the maximum mountain for a given form of the force, $f_i$.
- We calculate stars A and C – paying close attention to the traction conditions for star C.
- We normalise the amplitude by ensuring that star C reaches breaking strain at some point in the crust. (Note that this is the same calculation done by HJA, with a subtly different approach to the deforming force.)
- We also make sure that star A has the same force amplitude, so both stars experience the same $f_i$.

# Examples

# The stellar models

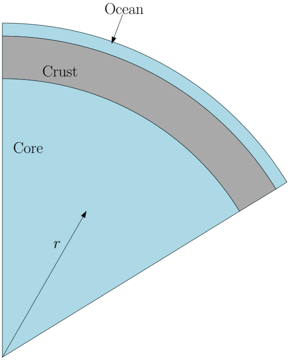
- As a proof of principle, we consider three examples for the deforming force. We do this to show how one can calculate mountains using this approach and to see how close we can get to the previous maximum mountain estimates.
- We use a polytropic equation of state for the spherical background,

$$p(\rho) = K\rho^{1+1/n},$$

with $n = 1$ and generate models with $M = 1.4\,M_\odot$, $R = 10\,\mathrm{km}$.
- For the shear-modulus profile in the crust, we have

$$\mu(\rho) = \kappa\rho,$$

where $\kappa = 10^{16}\,\mathrm{cm}^2\,\mathrm{s}^{-2}$.




Cross section of star C.

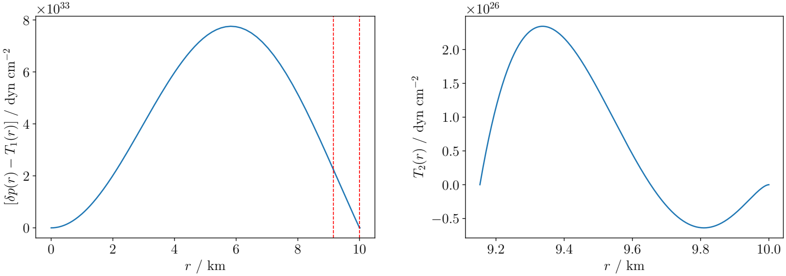
# A solution of Laplace's equation I

- The force has the form, $f_i = -\rho\nabla_i\chi$. The source potential is a solution of Laplace's equation,

$$\nabla^2\chi = 0.$$

- With regularity, this has the general solution,

$$\chi(r) = Ar^l.$$




The radial (left panel) and tangential (right panel) components of the perturbed traction for the potential solution to Laplace's equation.

# A solution of Laplace's equation II

- When the crust breaks we found,

$$|Q_{22}^{\mathrm{A}}| = 2.4 \times 10^{43} \left( \frac{\bar{\sigma}_{\mathrm{max}}}{10^{-1}} \right) \mathrm{g\,cm^2}, \quad |\epsilon^{\mathrm{A}}| = 3.1 \times 10^{-2} \left( \frac{\bar{\sigma}_{\mathrm{max}}}{10^{-1}} \right)$$

  and

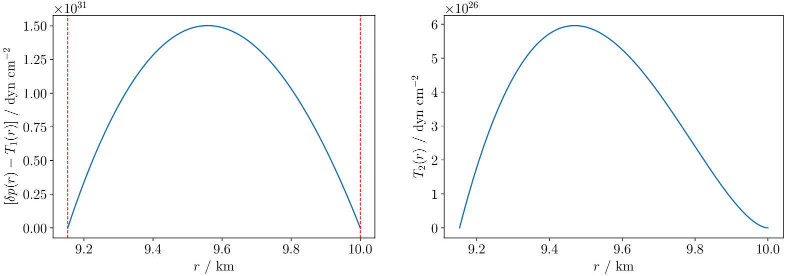
$$|Q_{22}^{\mathrm{C}} - Q_{22}^{\mathrm{A}}| = 1.7 \times 10^{37} \left( \frac{\bar{\sigma}_{\mathrm{max}}}{10^{-1}} \right) \mathrm{g\,cm^2}, \quad |\epsilon^{\mathrm{C}} - \epsilon^{\mathrm{A}}| = 2.2 \times 10^{-8} \left( \frac{\bar{\sigma}_{\mathrm{max}}}{10^{-1}} \right).$$

- This is about two orders of magnitude below previous calculations.

# A solution of Laplace's equation outside the core I

- In this case, the potential has the form,

$$\chi(r) = Ar^l + B/r^{l+1}.$$




The radial (left panel) and tangential (right panel) components of the perturbed traction for the potential solution to Laplace's equation that does not act in the core.

# A solution of Laplace's equation outside the core II

- We calculated,

$$|Q_{22}^{\mathrm{A}}| = 1.4 \times 10^{41} \left(\frac{\bar{\sigma}_{\max}}{10^{-1}}\right) \mathrm{g\,cm^2}, \quad |\epsilon^{\mathrm{A}}| = 1.8 \times 10^{-4} \left(\frac{\bar{\sigma}_{\max}}{10^{-1}}\right)$$

and

$$|Q_{22}^{\mathrm{C}} - Q_{22}^{\mathrm{A}}| = 4.4 \times 10^{38} \left(\frac{\bar{\sigma}_{\max}}{10^{-1}}\right) \mathrm{g\,cm^2}, \quad |\epsilon^{\mathrm{C}} - \epsilon^{\mathrm{A}}| = 5.7 \times 10^{-7} \left(\frac{\bar{\sigma}_{\max}}{10^{-1}}\right).$$

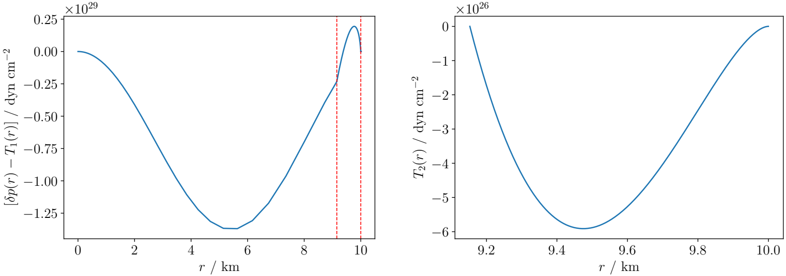
- This is a factor of a few below previous estimates and illustrates how these estimates depend on the prescription of the force.

# A thermal pressure perturbation I

- Introduce a thermal-pressure-like force,

$$f_i = -\nabla_i \delta p_{\text{th}} = -\frac{k_B}{m_u} \nabla_i (\rho \delta T),$$

and assume the perturbed temperature, $\delta T$, to be quadratic in form.




The radial (left panel) and tangential (right panel) components of the perturbed traction for the thermal pressure perturbation.

# A thermal pressure perturbation II

- We obtained,

$$|Q_{22}^{\mathrm{A}}| = 9.2 \times 10^{38} \left( \frac{\bar{\sigma}_{\mathrm{max}}}{10^{-1}} \right) \ \mathrm{g\,cm^2}, \quad |\epsilon^{\mathrm{A}}| = 1.2 \times 10^{-6} \left( \frac{\bar{\sigma}_{\mathrm{max}}}{10^{-1}} \right)$$

  and

$$|Q_{22}^{\mathrm{C}} - Q_{22}^{\mathrm{A}}| = 4.0 \times 10^{38} \left( \frac{\bar{\sigma}_{\mathrm{max}}}{10^{-1}} \right) \ \mathrm{g\,cm^2}, \quad |\epsilon^{\mathrm{C}} - \epsilon^{\mathrm{A}}| = 5.2 \times 10^{-7} \left( \frac{\bar{\sigma}_{\mathrm{max}}}{10^{-1}} \right).$$

- This is the same order of magnitude to the solution of Laplace's equation outside the core.

# Summary & future work

# Summary & future work I

- There are issues with previous calculations that made it worthwhile returning to the problem of constructing mountains on neutron stars.
- We introduce a new scheme to calculate mountains that explicitly satisfies the necessary boundary conditions. However, this scheme requires the introduction of a deforming force.
- We considered three examples and obtained maximum quadrupoles between a factor of a few to two orders of magnitude below previous estimates.

| Source | $\lvert Q_{22}^{\mathrm{A}} \rvert$ / g cm$^2$ | $\lvert \epsilon^{\mathrm{A}} \rvert$ | $\lvert Q_{22}^{\mathrm{C}} - Q_{22}^{\mathrm{A}} \rvert$ / g cm$^2$ | $\lvert \epsilon^{\mathrm{C}} - \epsilon^{\mathrm{A}} \rvert$ |
|---|---|---|---|---|
| Solution of Laplace's equation | $2.4 \times 10^{43}$ | $3.1 \times 10^{-2}$ | $1.7 \times 10^{37}$ | $2.2 \times 10^{-8}$ |
| Solution of Laplace's equation (outside core) | $1.4 \times 10^{41}$ | $1.8 \times 10^{-4}$ | $4.4 \times 10^{38}$ | $5.7 \times 10^{-7}$ |
| Thermal pressure perturbation | $9.2 \times 10^{38}$ | $1.2 \times 10^{-6}$ | $4.0 \times 10^{38}$ | $5.2 \times 10^{-7}$ |

- Such a force will depend on the (possibly quite complex) formation history of the star. For this reason, we believe that evolutionary calculations will be necessary to make progress on this problem (Bildsten, 1998; Singh *et al.*, 2020; Osborne and Jones, 2020).

# Summary & future work II

- We have followed the usual assumption that the crust can be well described as an elastic solid until it reaches breaking strain, at which point the crust fails and the strain is released. Typically, laboratory materials exhibit some plastic deformation before failure.
- A natural continuation of this work would be to extend this calculation to relativistic gravity. One would need to use the relativistic perturbation equations (Gittins, Andersson, and Pereira, 2020).

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